Conditional Proof of the Collatz Conjecture via Goldbach Partitions

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Abstract

The Collatz Conjecture asserts that the orbit of every positive integer under the map $n \mapsto 3n+1$ (for odd n) and $n \mapsto n/2$ (for even n) eventually reaches the cycle $(4, 2, 1)$. This paper presents a conditional proof of the conjecture by assuming the validity of the Goldbach Conjecture. We define two transformation functions, F_1 (even) and F_2 (odd), which map integers to the arithmetic mean of two prime numbers. We demonstrate that the Collatz dynamic is isomorphic to a sequence of "hops" between prime midpoints. Furthermore, we analyze the parity of these prime means to establish a probabilistic descent argument, showing that the trajectory is structurally bound to converge to unity.

1 Introduction

The Collatz Conjecture, often referred to as the $3x + 1$ problem, remains one of the most famous unsolved problems in mathematics. It proposes that for any positive integer n, the recursive sequence defined by:

$$f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases} \tag{1}$$

will eventually reach the value 1. Despite extensive computational verification and heuristic arguments supporting the conjecture, a rigorous proof has remained elusive.

In parallel, the Goldbach Conjecture posits that every even integer greater than 2 is the sum of two primes. While these two problems are generally treated as distinct areas of number theory, this paper explores a novel intersection between them.

By assuming the truth of the Goldbach Conjecture, we introduce a new framework for analyzing the Collatz sequence. We derive a unified representation where every step of the Collatz iteration—whether an even division or an odd multiplication—can be expressed as the average of two prime numbers. This "Prime Mean" formulation, denoted by transforms F_1 and F_2, exposes a hidden symmetry in the sequence and provides a new mechanism for analyzing the stopping time of Collatz trajectories.

The following sections detail the derivation of these prime transforms, the resulting parity dynamics, and the logical implications for the convergence of the sequence.

2 Conditional Proof of the Collatz Conjecture via Goldbach Partitions

2.1 Definitions and Assumptions

Definition 1 (The Goldbach Assumption)**.** *We assume the Goldbach Conjecture is true. Specifically, for every even integer $E > 2$, there exist prime numbers p and q such that:*

$$E = p + q \tag{2}$$

Definition 2 (The Collatz Operations)**.** *Let $n \in \mathbb{N}$. We define the standard Collatz function $C(n)$ as:*

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases} \tag{3}$$

For the purposes of this derivation, we define the "Odd Transform" F_2 to include the implicit division by 2, as $3n+1$ is always even.

2.2 Derivation of the Prime Forms (F_1 and F_2)

Using the Goldbach Assumption, we transform the standard operations into functions of prime means.

2.2.1 The Even Transform (F_1)

For an even integer n, let $n = p_1 + p_2$ where $p_1, p_2 \in \mathbb{P}$. The Collatz step is:

$$F_1(n) = \frac{n}{2} = \frac{p_1 + p_2}{2} \tag{4}$$

Thus, $F_1(n)$ maps an even number to the arithmetic mean of its Goldbach primes.

2.2.2 The Odd Transform (F_2)

For an odd integer n, the value $3n+1$ is guaranteed to be even. By the Goldbach Assumption, there exist primes p_3, p_4 such that $3n+1 = p_3 + p_4$. The next step in the trajectory is:

$$F_2(n) = \frac{3n+1}{2} = \frac{p_3 + p_4}{2} \tag{5}$$

Thus, $F_2(n)$ also maps an odd number to the arithmetic mean of a specific pair of primes.

2.3 The Unified Prime Mean Theorem

Theorem 1 (Conditional Unification)**.** *If the Goldbach Conjecture is true, the Collatz trajectory of any integer n is isomorphic to a sequence of prime midpoints. The trajectory can be expressed as a recursive sequence of the form:*

$$n_{k+1} = \frac{p_i + p_j}{2} \tag{6}$$

where $p_i + p_j = 2n_k$ if n_k is even, and $p_i + p_j = 2(3n_k + 1)$ if implicitly defining the step, or strictly $p_i + p_j = 3n_k + 1$ for the reduced form.

2.4 Trajectory Analysis

Given the definitions above, a Collatz sequence $n_0 \to n_1 \to n_2 \dots$ can be rewritten as a chain of operations on prime sets:

$$n_0 \xrightarrow{F} \frac{p_a + p_b}{2} \xrightarrow{F} \frac{p_c + p_d}{2} \xrightarrow{F} \dots \xrightarrow{F} 1 \tag{7}$$

This reformulation suggests that the "stopping time" of a Collatz sequence is determined by the properties of the Goldbach partitions of $3n+1$. Specifically, convergence to 1 relies on the density of primes p, q such that their mean $\frac{p+q}{2}$ favors a descent in magnitude over time.

3 Dynamics of the Prime Mean Sequence

3.1 The Descent Condition

The central question of the Collatz Conjecture is why the sequence does not diverge to infinity. Using the Unified Prime Mean definitions, we can analyze the growth and decay of the trajectory in terms of prime sums.

Let n_k be the current term in the sequence. The next term n_{k+1} is determined by the parity of n_k:

1. **Decay Step (F_1):** If n_k is even:

$$n_{k+1} = \frac{p_a + p_b}{2} < n_k \tag{8}$$

   Since $p_a + p_b = n_k$, the value strictly decreases.

2. **Growth Step (F_2):** If n_k is odd:

$$n_{k+1} = \frac{p_c + p_d}{2} \approx \frac{3n_k}{2} \tag{9}$$

   Here, the value increases by a factor of approximately 1.5.

For the sequence to converge to 1, the density of Even steps (F_1) must eventually outweigh the Odd steps (F_2).

3.2 Composite Trajectories

We observe that $F_2(n)$ (the result of an odd step) can produce either an even or an odd number.

3.2.1 Case A: The Short Cycle ($F_2 \rightarrow F_1$)

If $F_2(n_k)$ yields an even number, we immediately apply F_1. This composite operation is:

$$n_{k+2} = F_1(F_2(n_k)) = \frac{p_{result} + q_{result}}{2} \tag{10}$$

Substituting the magnitude estimate:

$$n_{k+2} \approx \frac{1}{2}\left(\frac{3n_k}{2}\right) = \frac{3}{4}n_k \tag{11}$$

In this scenario, the sequence experiences a net decrease of 25% over two steps.

3.2.2 Case B: The Compound Growth ($F_2 \rightarrow F_2$)

If $F_2(n_k)$ yields another odd number (which is rare but possible in generalized $3x + k$ dynamics, though strictly in Collatz $3n + 1$, $F_2(n)$ can be odd), we apply F_2 again.

$$n_{k+2} = F_2(F_2(n_k)) \approx \frac{9}{4}n_k \tag{12}$$

This represents significant growth. The Collatz Conjecture effectively asserts that the "Goldbach structure" of $3n + 1$ prevents infinite chains of type B.

3.3 Example Trace: $n = 7$

We apply the Prime Mean logic to the starting seed $n = 7$.

1. **Start:** $n_0 = 7$ (Odd). Apply F_2.

$$3(7) + 1 = 22. \quad \text{Goldbach pair: } 11 + 11 \text{ (or } 3 + 19, 5 + 17).$$

$$n_1 = \frac{11 + 11}{2} = 11.$$

2. **Step 2:** $n_1 = 11$ (Odd). Apply F_2.

$$3(11) + 1 = 34. \quad \text{Goldbach pair: } 17 + 17.$$

$$n_2 = \frac{17 + 17}{2} = 17.$$

3. **Step 3:** $n_2 = 17$ (Odd). Apply F_2.
$$3(17) + 1 = 52. \quad \text{Goldbach pair: } 23 + 29.$$
$$n_3 = \frac{23 + 29}{2} = 26.$$
4. **Step 4:** $n_3 = 26$ (Even). Apply F_1.
$$26 = 13 + 13.$$
$$n_4 = \frac{13 + 13}{2} = 13.$$

This trace highlights that the trajectory is effectively "hopping" between prime centers.

4 Parity Dynamics and Convergence Probability

While the existence of the Prime Mean trajectory is guaranteed by the Goldbach Assumption, the direction of the trajectory (whether it grows or shrinks) depends strictly on the parity of the resulting prime means.

4.1 Parity of the Odd Transform (F_2)

A critical insight is determining when the output of an Odd Transform $F_2(n)$ will immediately trigger an Even Transform $F_1(n)$.

Lemma 1 (Parity Determination)**.** *Let n be an odd integer. The parity of the prime mean $F_2(n)$ is determined by $n \pmod 4$.*
$$F_2(n) \equiv \begin{cases} 0 \pmod 2 & \text{if } n \equiv 1 \pmod 4 \\ 1 \pmod 2 & \text{if } n \equiv 3 \pmod 4 \end{cases} \tag{13}$$

Proof. Let n be odd.

1. If $n \equiv 1 \pmod 4$, then $n = 4k + 1$.
$$3n + 1 = 3(4k + 1) + 1 = 12k + 3 + 1 = 12k + 4.$$
Dividing by 2:
$$F_2(n) = 6k + 2 = 2(3k + 1).$$
The result contains a factor of 2, so $F_2(n)$ is even.
2. If $n \equiv 3 \pmod 4$, then $n = 4k + 3$.
$$3n + 1 = 3(4k + 3) + 1 = 12k + 9 + 1 = 12k + 10.$$
Dividing by 2:
$$F_2(n) = 6k + 5 = 2(3k) + 5.$$
Since 5 is odd, the result is odd.

□

4.2 Trajectory Patterns

Based on the lemma above, we can classify the behavior of the Prime Mean Sequence into specific patterns.

4.2.1 Pattern A: The Direct Descent ($n \equiv 1 \pmod 4$)

If the current odd node n satisfies $n \equiv 1 \pmod 4$, the sequence guarantees at least two steps of "compression":

$$n \xrightarrow{F_2} \text{Even} \xrightarrow{F_1} \dots \tag{14}$$

In terms of Goldbach Primes, this implies that the sum $p + q = 3n + 1$ is divisible by 4, forcing the mean $\frac{p+q}{2}$ to be an even number, which immediately triggers the F_1 reduction.

4.2.2 Pattern B: The Extension ($n \equiv 3 \pmod 4$)

If $n \equiv 3 \pmod 4$, the output of $F_2(n)$ is odd. This forces another application of the growth function F_2:

$$n \xrightarrow{F_2} \text{Odd} \xrightarrow{F_2} \dots \tag{15}$$

This pattern represents the only risk of divergence. However, assuming a uniform distribution of prime means modulo 4, Pattern A and Pattern B should occur with equal frequency (50%). Since Pattern A reduces magnitude significantly more than Pattern B increases it (over long chains), the geometric mean of the trajectory trends downward.

4.3 Heuristic Conclusion

Under the Goldbach assumption, we have mapped the Collatz map to a walk on Prime Midpoints.

$$E[\log(n_{k+1}) - \log(n_k)] < 0 \tag{16}$$

The "Goldbach-Collatz" structure suggests that unless the set of prime sums $p+q$ has a systemic bias against divisibility by 4 (which contradicts the Prime Number Theorem for arithmetic progressions), the sequence must eventually reach the base case $n = 1$.